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Use of Exponentials in the Integral Solution of the Parabolic Equations of Boundary Layer, Wake, Jet, and Vortex Flows

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The equations of incompressible laminar boundary layer, vortex, and axisymmetric wake/jet flows are solved by the method of weighted residuals using exponentials in both approximating and weighting functions. All formulations are written for arbitrary unspecified order N of approximation. Two particular formulations of velocity and circulation profiles are used—one containing a power series modified by an exponential, the other a series of exponentials. The exponential series method is found to be superior to the power series formulation with respect to convergence, reliability, and ease of application. It produces accurate and apparently convergent results for relatively small computing effort. Results are presented for the transition of the asymptotic suction to the Blasius profile, for the circular cylinder, and for typical vortex flows. Extensions to compressible and turbulent flows are possible.

I. INTRODUCTION

A large and important group of two- and three-dimensional fluid dynamic problems is governed by parabolic approximations to the Navier–Stokes equations in two independent variables. Among these are the plane boundary layer flow, wake, and jet, the yawed (three-dimensional) boundary layer, wake, and jet, and the axisymmetric boundary layer on a body of finite radius (by Mangler's transformation), the axisymmetric wake, jet, and boundary layer on a needle, the quasi-cylindrical vortex, and the boundary layer on a spinning needle (see Table I and Fig. 1).

Methods of weighted residuals based on the work of Galerkin [1] and Kantorovich [2] (here briefly "integral methods") for the computation of some of these flows will be described. The laminar incompressible case will be considered throughout, but the methods can be extended to compressible and/or turbulent flows.

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TABLE I

Flows Described by Boundary-Layer Type Partial Differential Equations in Two Coordinates

		Boundary Layer	Wake/Jet
Cartesian Case	2-dimensional velocity vector $u = u(x, y)$ $v = v(x, y)$	Plane boundary layer $u(x, 0) = 0$ $v(x, 0) = v_0(x)$ by Mangler transformation 3-dimensional axisymmetric boundary layer for $r \gg \delta$	Plane wake/jet $u_y(x, 0) = 0$ $v(x, 0) = 0$
	3-dimensional velocity vector $u = u(x, y)$ $v = v(x, y)$ $w = w(x, y)$	Boundary layer on yawed infinite cylinder $u(x, 0) = 0$ $v(x, 0) = v_0(x)$ $w(x, 0) = 0$	Yawed infinite wake/jet $u_y(x, 0) = 0$ $v(x, 0) = 0$ $w_y(x, 0) = 0$
Axisymmetric Case	2-dimensional velocity vector $u = u(x, r)$ $v = v(x, r)$	Boundary layer on needle $u(x, 0) = 0$ $v(x, 0) = v_0(x)$	Axisymmetric wake/jet $u_r(x, 0) = 0$ $v(x, 0) = 0$
	3-dimensional velocity vector $u = u(x, r)$ $v = v(x, r)$ $w = w(x, r)$	Boundary layer on spinning needle $u(x, 0) = 0$ $v(x, 0) = v_0(x)$ $w(x, 0) = 0$	Quasicylindrical vortex $u_r(x, 0) = 0$ $v(x, 0) = 0$ $w(x, 0) = 0$

Finite difference methods for the accurate calculation of these parabolic flows are well-developed. Yet the prospect of greater computing economy has kept interest in integral methods alive, if just barely. It is well to recall some shortcomings of traditional integral methods, such as the Karman–Pohlhausen [3] and Walz and Thwaites methods [4, 5]:

Limitation to one-parameter family or similarity profiles;

Inadequate accuracy of local results (the velocity profile), while overall results (wall shear, displacement and momentum thicknesses) are usually adequately described;

Use of empirical constants;

Integration in a region with vaguely defined upper boundary δ (boundary layer thickness).

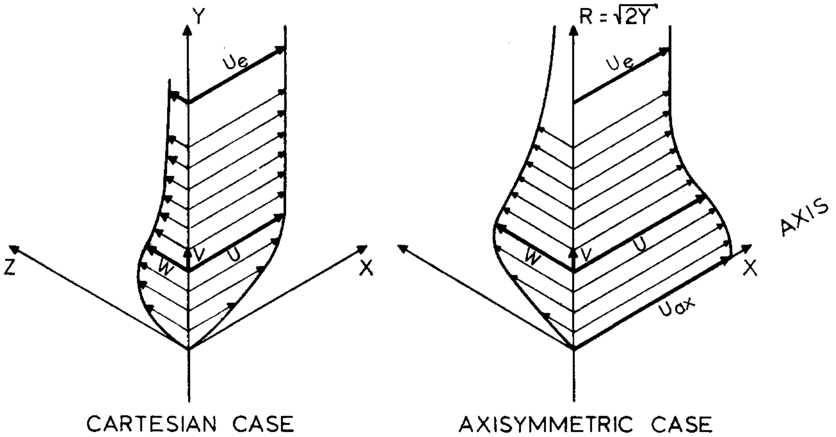

FIG. 1. Coordinates and velocities.

In the past decade, Dorodnitsyn [6] and his coworkers (see survey in [7]) have formulated integral methods for boundary layer flows which do not suffer the shortcomings cited above and which have put integral methods into direct competition with finite difference methods. Bethel [8, 9] has thoroughly investigated the Dorodnitsyn method by computing many test cases and comparing them to exact results obtained by other methods. He was able to show that in most cases the method appears to be converging to the correct solution as the number of parameters in the approximating function is increased. However, in a few cases the method evidently converges to an incorrect solution. Some objections to the method can be raised:

By using the velocity u as independent variable, the shear profile is approximated by powers of u. It is questionable whether powers of some velocity profiles (e.g., separation profile) will correspond to a complete set yielding uniform approximation of the shear profile.

Different approximating functions must be used in regions of accelerated and retarded flow.

The formulation cannot handle velocity overshoot (in one integration).

Special methods are required for dealing with elliptic integrals occurring for $N \geqslant 4$. In the case of retarded flow numerical integration becomes necessary.

The excellent results obtained by the Dorodnitsyn method encouraged a search for related methods which would not have the shortcomings just listed. The ultimate aim must be the development of integral methods yielding solutions converging to the exact solution as the number of parameters is increased, and giving reliable engineering estimates for low orders of approximation and small investment in computing time.

Results of studies of integral methods for boundary layer and vortex flows, and related flows where flow variables are functions of only two independent variables, are reported here. The exponential character of the distributions of the independent variables through the shear regions of the flows considered leads to the adoption of approximating functions containing exponentials. On mathematical grounds, two particular formulations appear particularly well-suited, and are considered further. They lead to two distinct integral methods, one using a power series modified by an exponential, the other a series of exponentials. Results of these methods for boundary layer, wake, jet, and vortex flows will be presented. Best results are obtained with the exponential series method.

II. General Method and General Formulae

The general integral relations for incompressible plane boundary and wake/jet flows and for vortex and axisymmetric wake/jet flows will first be derived without reference to any particular approximating or weighting functions. Specific approximating and weighting functions will be introduced in the following section.

In the boundary layer case it is clear that transformations such as Mangler's and standard compressibility transformations can be applied to obtain corresponding axisymmetric and compressible flows. This will not be pursued further.

A. *Plane Boundary Layer and Wake/Jet Flow*

The nondimensional incompressible boundary layer equations are, in divergence form,

$$\frac{\partial(U^2)}{\partial X} + \frac{\partial(VU)}{\partial Y} = U_e \frac{dU_e}{dX} + \frac{\partial^2 U}{\partial Y^2}$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0.$$

The boundary conditions for the boundary layer case are

$$U(X, 0) = 0 \qquad U(X, \infty) = U_e(X)$$

$$V(X, 0) = V_0(X) \qquad U(0, Y) = \text{initial profile.}$$

In the case of the plane wake/jet the boundary conditions are

$$\frac{\partial U}{\partial Y}(X, 0) = 0 \text{ (symmetry)} \qquad U(X, \infty) = U_e(X)$$

$$V(X, 0) = 0 \qquad U(0, Y) = \text{initial profile.}$$

Nondimensional and physical variables are related by

$$X = x/l, \qquad U = u/u_\infty,$$
$$Y = \sqrt{Re}\, y/l, \qquad V = \sqrt{Re}\, v/u_\infty,$$

where $Re = u_\infty l/\nu$.

It is often desirable, for numerical or analytic reasons, to keep variables in the flow calculation of approximately constant magnitude (or exactly constant in similarity cases). A transformation will now be introduced which scales the independent variable Y according to a prescribed function $g(X)$. It should be noted that this transformation is introduced solely for computational convenience. The choice of $g(X)$ is not critical, and in fact in many cases (e.g., circular cylinder) $g(X) = 1$ is a perfectly satisfactory choice.

Introduce the transformed coordinates

$$\bar{Y} = \bar{Y}/g(X) \qquad \bar{X} = X$$

and the transformed velocities

$$\bar{U} = Ug \qquad \bar{V} = V - \bar{Y}\frac{dg}{dX}U.$$

The transformed nondimensional incompressible plane boundary layer equations then become

$$\frac{\partial(\bar{U}^2)}{\partial\bar{X}} + \frac{\partial(\bar{U}\bar{V})}{\partial\bar{Y}} = \bar{U}^2\frac{dg/dX}{g} + g^2U_e\frac{dU_e}{dX} + \frac{1}{g}\frac{\partial^2\bar{U}}{\partial\bar{Y}^2}$$
$$\frac{\partial\bar{U}}{\partial\bar{X}} + \frac{\partial\bar{V}}{\partial\bar{Y}} = 0.$$

The boundary conditions transform correspondingly.

To obtain the general integral relations, the momentum equation is multiplied by members of a set of linearly independent weighting functions $f_k(\bar{Y})$. Convergence of the integrals requires $f_k(0) =$ finite and $f_k(\infty) = 0$.

The integral relations become

$$\frac{d}{d\bar{X}}\int_0^\infty f_k\bar{U}^2\,d\bar{Y} - \int_0^\infty f_k'\bar{V}\bar{U}\,d\bar{Y} - \frac{1}{g}\int_0^\infty f_k''\bar{U}\,d\bar{Y} + \bar{T}_0$$
$$- U_e\frac{dU_e}{dX}g^2\int_0^\infty f_k\,d\bar{Y} - \frac{dg/dX}{g}\int_0^\infty f_k\bar{U}^2\,d\bar{Y} = 0 \qquad (1)$$

For the boundary layer case,

$$\bar{T}_0 = \left[\frac{f_k}{g}\frac{\partial\bar{U}}{\partial\bar{Y}}\right]_{\bar{Y}=0},$$

and for the plane wake/jet,

$$\bar{T}_0 = -\left[\frac{f_k'\bar{U}}{g}\right]_{\bar{Y}=0}$$

The integral relations can be integrated in the $\bar{Y}$ direction once choices of weighting functions $f_k(\bar{Y})$ and of the velocity approximation $\bar{U}(\bar{X}, \bar{Y})$ have been made. The integral relations then reduce to a set of ordinary differential equations for the $\bar{X}$-dependent parameters in the velocity approximation. The velocity $\bar{V}(\bar{X}, \bar{Y})$ follows from the continuity equation. For the boundary layer, with prescribed suction or blowing velocity $V_0(X)$,

$$\bar{V}(\bar{X}, \bar{Y}) = -\int_0^{\bar{Y}} \frac{\partial \bar{U}}{\partial \bar{X}}\, d\bar{Y} + V_0(X)$$

and for the plane wake/jet:

$$\bar{V}(\bar{X}, \bar{Y}) = -\int_0^{\bar{Y}} \frac{\partial \bar{U}}{\partial \bar{X}}\, d\bar{Y}.$$

Alternately a streamfunction could have been introduced, but the required amount of algebra is equivalent to that of the present formulation.

B. *Vortex and Axisymmetric Wake/Jet*

The axisymmetric wake and jet are special cases of quasicylindrical vortex flow. Corresponding solutions are obtained by specifying zero swirl in the vortex formulation. With this in mind, only equations for the more general vortex flow case will now be developed.

With the coordinates and velocities of Fig. 1 the dimensional equations describing incompressible quasicylindrical vortex flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0,$$

$$\frac{w^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r},$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right),$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial r} + \frac{vw}{r} = \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w^2}{r^2}\right).$$

The equations are simplified by introduction of the following transformation of variables:

$$y = r^2/2,$$
$$h = vr,$$
$$k = wr \quad \text{(circulation)}.$$

In addition the equations are nondimensionalized by introduction of a Reynolds number $Re = u_\infty r_c/\nu$ based on freestream (axial) velocity and a representative vortex core radius r_c. We use

$$
\begin{aligned}
&X = x/r_c, \\
&R = \sqrt{Re}\,(r/r_c), \qquad Y = Re(y/r_c^2), \\
&U = u/u_\infty, \\
&V = \sqrt{Re}\,(v/u_\infty), \quad H = Re(h/u_\infty r_c), \\
&W = w/u_\infty, \qquad\quad K = \sqrt{Re}\,(k/u_\infty r_c). \\
&P = p/(\rho u_\infty^2/2),
\end{aligned}
$$

The nondimensional vortex equations then become

$$
\begin{aligned}
&\frac{\partial U}{\partial X} + \frac{\partial H}{\partial Y} = 0, \\
&\frac{K^2}{4Y^2} = \frac{\partial P}{\partial Y}, \\
&U\frac{\partial U}{\partial X} + H\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + 2\frac{\partial}{\partial Y}\left(Y\frac{\partial U}{\partial Y}\right), \\
&U\frac{\partial K}{\partial X} + H\frac{\partial K}{\partial Y} = 2Y\frac{\partial^2 K}{\partial Y^2},
\end{aligned}
$$

with the boundary conditions

$$
\begin{aligned}
&\frac{\partial U}{\partial Y}(X, 0) = 0 \text{ (symmetry)}, \qquad && U(X, \infty) = U_e(X), \\
&H(X, 0) = 0, && U(0, Y) = \text{initial profile}, \\
&K(X, 0) = 0, && K(X, \infty) = K_e(X), \\
& && K(0, Y) = \text{initial profile}.
\end{aligned}
$$

The pressure P is eliminated by cross-differentiation, and H is found by formal integration from the continuity equation,

$$H = -\int_0^Y \frac{\partial U}{\partial X}\,dY.$$

Two equations remain

$$\frac{\partial}{\partial X}(UK) + \frac{\partial}{\partial Y}\left(HK - 2Y\frac{\partial K}{\partial Y} + 2K\right) = 0$$

and

$$\frac{\partial}{\partial X}\left(2Y^2U\frac{\partial U}{\partial Y} + \frac{K^2}{4}\right) + Y^2\frac{\partial^2}{\partial Y^2}\left(HU - 2Y\frac{\partial U}{\partial Y}\right) = 0.$$

The last equation has been multiplied by Y^2. (This amounts to applying the weighting functions $\{Y^2 f_k(Y)\}$). Weighting functions $g_k(Y)$ and $f_k(Y)$ are now introduced with the requirements $g_k(0) = \text{finite}$, $g_k(\infty) = 0$; $f_k(0) = \text{finite}$, $f_k(\infty) = 0$. Multiplication of the two equations by g_k and f_k, respectively, and integration results in the two integral relations,

$$\frac{d}{dX}\int_0^\infty g_k UK\,dY - \int_0^\infty g_k' HK\,dY - \int_0^\infty (2g_k'' Y + 4g_k')\,K\,dY = 0$$

and

$$\begin{aligned}\frac{d}{dX}\int_0^\infty (f_k' Y^2 + 2f_k Y)\,U^2\,dY &- \frac{d}{dX}\int_0^\infty f_k \frac{K^2}{4}\,dY \\ &- \int_0^\infty (f_k'' Y^2 + 4Y f_k' + 2f_k)\,HU\,dY \\ &- \int_0^\infty (2f_k''' Y^3 + 14 f_k'' Y^2 + 20 f_k' Y + 4f_k)\,U\,dY = 0. \qquad (2)\end{aligned}$$

After introduction of weighting functions f_k and g_k and of approximating expressions for velocity and circulation profiles $U(X, Y)$ and $K(X, Y)$ two sets of ordinary differential equations are obtained for the X-dependent parameters of the velocity and circulation approximations.

In the vortex case a pseudosimilarity transformation is not required because the vortex core remains of approximately constant diameter in the quasicylindrical case for which these equations are valid.

C. *Determination of Parameters*

We are here interested in integral methods where the unknown parameters in the approximating expressions are all or in part determined by weighted residuals. Remaining parameters may be found from applying compatibility conditions at the boundaries, as in the Karman-Pohlhausen method. The use of compatibility conditions should be carefully considered, since they may have catastrophic effects on the total flow where boundary conditions are discontinuous (e.g., suction). For this reason all parameters in the present methods are determined by weighted residuals.

III. Approximation Using Exponentials: Initial Profiles

A. *Choice of Weighting and Approximating Functions*

Suitable weighting and approximating functions must now be chosen in order to permit analytical integration, with respect to Y, of the integral relations derived in the previous section.

Weighting functions chosen for the integral relations of the previous section must satisfy several requirements: (1) the members of each set $\{f_k(Y)\}$ and $\{g_k(Y)\}$ must be linearly independent; (2) the f_k and g_k must result in convergent integrals; (3) they should insure analytical integrability; (4) the functions should weight most heavily the region in which the solution is being sought, i.e., the shear region near $Y = 0$. The requirements are satisfied by the set $\{e^{-\sigma(k)Y}Y^m\}$ subject to a proper choice of approximating functions to satisfy the third condition. Note that the choice $\{e^{-kY}\}$ corresponds approximately to the weighting function set $\{(1 - u)^k\}$ used by Dorodnitsyn [6].

The choice of approximating functions is guided by the experimentally observed behavior of boundary layer, wake/jet, and vortex flows, and by a few exact solutions. All of these flows have an exponential character. It is thus logical to choose approximating functions containing exponentials. Extremely simple integrals of the form,

$$\int_0^\infty e^{-\alpha Y} Y^n \, dY = \frac{n!}{\alpha^{n+1}},$$

result from adopting elements of the set $\{e^{-\alpha Y}Y^n\}$ for use in the approximating functions also. We require expressions of the series type and have as the most general candidate,

$$\sum_n a_n e^{-\alpha_n Y} Y^n.$$

This expression could probably lead to good approximation using only very few terms, but its use requires more complicated algebra than the following two special cases:

$$e^{-\alpha Y} \sum_n a_n Y^n$$

and

$$\sum_n a_n e^{-n\alpha Y}.$$

These two—one involving a power series in Y, the other a series of exponentials in Y—appear more promising and will be used in the present work.

The first expression is nonlinear. While it appears to be capable of adequately representing most of the desired profiles, clearly the role of the exponent α is an important one, and the accuracy of the approximation for a given N, and possibly the convergence, will depend on it. In the light of this it is doubtful whether the set is complete for general α. The maxima of the functions in the set are at $Y = n/\alpha$. When α is small, it is unlikely that functions with rapid changes near $Y = 0$ can be properly approximated.

The second expression is linear in powers of $e^{-\alpha Y}$. If a coordinate transformation $\eta = e^{-\alpha Y}$ is introduced, it is clear that we are dealing with a polynomial expression in a finite interval $1 \geqslant \eta > 0$ (corresponding to $0 \leqslant Y < \infty$). The polynomial set is complete, and uniform approximation for any continuous function in [1, 0] follows by Weierstrass' theorem. Corresponding orthonormal functions are the shifted Legendre polynomials in [0, 1]. This observation is reassuring; however, it still does not prove convergence of the complete integral method.

In the following, integral methods using the two approximations above will be developed. The first will be referred to as the *power series method*, the second as *exponential series method*. Before we attempt to solve the flow equations, it is worthwhile to investigate the approximation characteristics of both methods. A procedure is required anyway for finding the initial parameters for given arbitrary initial profiles.

B. *Approximation by Weighted Residuals—General Method*

The N unknown coefficients in the velocity approximation are here determined by requiring the integral of the weighted difference (residual) between exact and approximate velocity profile over the region $(0, \infty)$ to vanish for N different choices of (linearly independent) weighting functions. In the case of the exponential series method with $f_k = e^{-k\alpha Y}$, this amounts to a least squares approximation in the region (0, 1) of the η domain. Thus, if $U(Y)$ is the exact, $U_N(Y; a_n)$ the approximate profile, and $f_k(Y)$ the weighting function, then the a_n are here obtained from

$$\int_0^\infty f_k(Y)[U(Y) - U_N(Y; a_n)]\, dY = 0, \qquad k = 1, 2,..., N. \tag{3}$$

(One more equation is required if α is an unknown parameter, as in the power series method as used here.) The quality of the approximation can be judged from the error distribution $U(Y) - U_N(Y) = \epsilon(Y)$. A single number is more convenient for an appraisal of error magnitude, such as the mean square error

$$E = \int_0^\infty \epsilon^2(Y)\, dY = \int_0^\infty [U(Y) - U_N(Y)]^2\, dY.$$

For a convergent method we must have $E \to 0$ as $N \to \infty$.

We shall now test the approximation methods by applying them to a ramp profile $[U = Y\ (0 \leqslant Y < 1)$ and $U = 1\ (Y > 1)]$, where analytical integration is possible, and the error E can be computed exactly. This will provide some insight into convergence properties of the approximation.

C. *Approximation by the Power Series Method*

The approximation for the boundary layer profile in the power series method is

$$U(Y) = U_e + e^{-\alpha Y} \sum_{n=0}^{N} a_n Y^n, \tag{4}$$

where α and the a_n are undetermined parameters with $a_0 = -U_e$. This formulation satisfies the boundary conditions and contains the asymptotic suction profile as an exact solution for $a_n = 0$, $n \geqslant 1$. Weighting functions $f_k(Y) = e^{-\sigma(k)y}$ were used in Eqs. (3). In this formulation a system of $(N + 1)$ nonlinear equations must be solved simultaneously for α and the N unknown parameters a_n.

Convergence properties are illustrated by the results of the analytical integrations for the ramp profile. Figure 2 shows the mean-square-error E as a function of the order of approximation N. Integer weighting function exponents $\sigma = 1, 2, \ldots, N + 1$ were used in these calculations, but the error E is hardly affected by the choice of weighting function exponents in the range from 0.1 to 10.

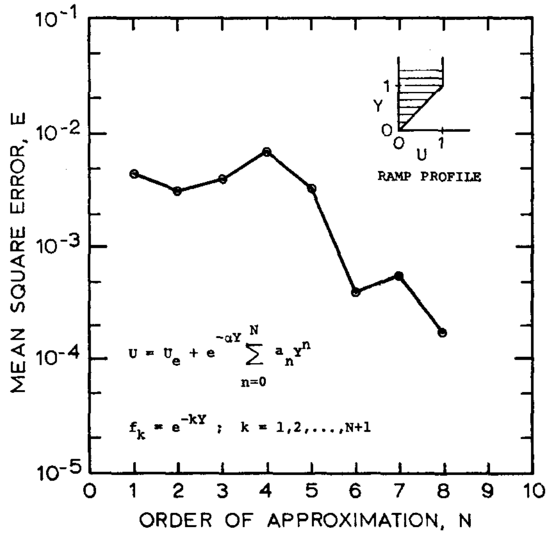

FIG. 2. Mean-square error in the approximation of a ramp profile by the power series method.

D. *Approximation by the Exponential Series Method*

In the exponential series method, the boundary layer velocity profile is approximated by

$$U(Y) = (1 - e^{-\alpha Y}) \left(U_e + \sum_{n=1}^{N} a_n e^{-n\alpha Y} \right), \tag{5}$$

where the a_n are the parameters to be determined, and α is a constant of the order of the exponent of the asymptotic suction profile having the same displacement thickness. The formulation satisfies the boundary conditions and contains the asymptotic suction profile as an exact solution for $a_n = 0$, $n \geqslant 1$. Again weighting functions $f_k(Y) = e^{-\sigma(k)Y}$ are used to obtain the N unknown parameters a_n from Eqs. (3). Substitution of the velocity approximation into Eqs. (3) leads to a least-squares approximation and a set of simultaneous linear equations which are solved for the a_n by Gaussian elimination.

The ramp case was again employed to confirm the theoretical convergence properties of the approximation by computing the mean-square-error E exactly. The results for $\alpha = 1$ are plotted in Fig. 3. Again integer weighting function exponents $\sigma = 1, 2, \ldots, N$ were used.

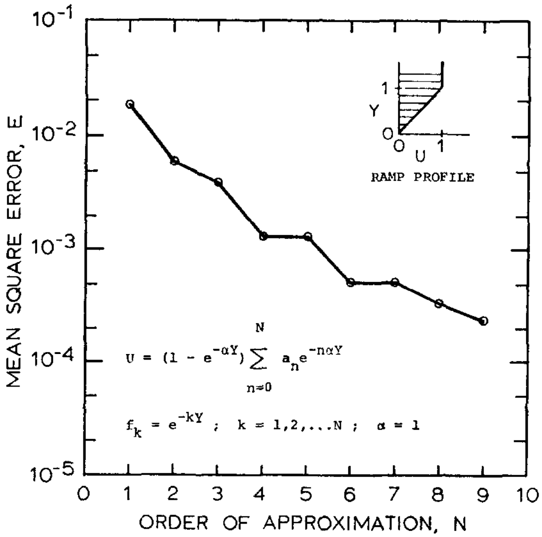

FIG. 3. Mean-square error in the approximation of a ramp profile by the exponential series method.

E. *Comparison and Discussion*

We compare the profile approximations by the power series method (Fig. 2) and the exponential series method (Fig. 3). Both methods are clearly capable of approximating the representative velocity profiles very well. However, the behavior of the power series method is more erratic than that of the exponential series approximation, and the occasional appearance of divergent solutions foreshadows similar problems in the full integral method. It will also later be shown that the full integral method using power series approximation generally fails to duplicate the local accuracy of the present profile approximation.

The power series and the exponential series approximations will now be introduced into the integral relations for the boundary layer and for vortex flows (with wakes and jets as special cases). All methods were developed, like the corresponding profile approximations by weighted residuals, for arbitrary unspecified order of approximation N.

IV. Application of the Power Series Method

A. *Boundary Layer Flow*

The velocity approximation in the power series method is (with $a_0 = -U_e$) in the boundary layer case, and $a_0(X)$ free in the wake–jet case.

$$U(X, Y) = U_e(X) + e^{[illegible]} \sum_{n=0}^{[illegible]} a_n(X) Y^{n}$$

Note again that here $\alpha(X)$, in addition to the $a_n(X)$, is one of the undetermined parameters. The weighting functions

$$f_k(Y) = e^{-\sigma(k)Y}, \qquad k = 1, 2, \ldots, N + 1,$$

are chosen as for the profile approximation. They satisfy the previously stated end conditions with $f_k(0) = 1$, and $f_k(\infty) = 0$.

Introduction of these functions into the general integral relations for the boundary layer yields a system of $(N + 1)$ first-order ordinary nonlinear differential equations for $\alpha(X)$ and the $a_n(X)$. The system is linear in the derivatives and can be solved by standard procedures such as the Runge–Kutta method. A computational singularity appears when $a_N \rightarrow 0$.

A typical—and disappointing—result of the method is that for the flat plate. Constant suction over an infinitely long plate is discontinued at $X = 0$, and the asymptotic suction profile should develop into the Blasius profile. Displacement thickness and wall-shear development are as expected and check with other solutions, but the computed profile definitely converges to a wrong solution. Higher orders of approximation yield practically the same result. If the computation is started with the Blasius profile as initial profile (parameters obtained by the methods of Section III) violent gradients appear in the first few steps and all parameters undergo drastic changes until the profile has again adjusted to the incorrect shape.

These results are distressing especially in view of the fact that the profile approximation by the power series method proved to be very accurate. For a reason which is not obvious, and in contrast to the corresponding profile approximation, the boundary layer profile is described mainly by the exponent $\alpha(X)$; that is the zeroth-order approximation of asymptotic suction type. $\alpha(X)$ has a correspondingly low value and the correction terms of order two and higher peak outside of the boundary layer (at $Y = n/\alpha$). It was noted in the section on profile approximation by the power series method that accurate approximation is then apparently not possible. By contrast in the initial profile approximation using the same functions, the resulting $\alpha(X)$ for the Blasius profile is much higher. It increases with N, giving maxima of the first ten or so correction terms within the boundary layer, i.e., in the region where they are needed most, and the approximation becomes very accurate. We conclude again that the approximating set is not complete and that the quality of the approximation is a function of the exponent $\alpha(X)$. A consequence is that it should be possible to improve the approximation by forcing a large α, perhaps by simply keeping it constant at a sufficiently high value. A related possibility will now be discussed. It might be added that weighting function effects were studied by using weighting function exponents in the range $0.01 \leqslant \sigma \leqslant 10$, and by using the set $Y^2 e^{-\sigma(k)Y}$. No improvement could be obtained. Extreme σ can be expected to produce poor results if the region of interest is not adequately "covered."

The possibility of fixed α has been explored by Devan [10]. Here we will now keep a free $\alpha(X)$ in the dominant term of the velocity approximation, while using a constant, and preselected, exponent γ on all higher order terms. Thus, we use the velocity approximation,

$$U(X, Y) = U_e(X)(1 - e^{-\alpha(X)Y}) + e^{-\gamma Y} \sum_{n=1}^{N} a_n(X)\, Y^n,$$

retaining the same weighting functions as before, i.e., $f_k(Y) = e^{-\sigma(k)Y}$. The transition from the asymptotic suction to the Blasius case was again investigated. Computations with low (<2) and high (>3) values for γ broke down relatively soon. Long runs and accurate results were obtained for $\gamma \approx 2.5 (N = 3)$, see Fig. 4.

Some conclusions can be drawn. The integral solution based on the power series velocity approximation can be useful and accurate, but only if exponents α (or γ) are relatively high. Unfortunately, the power series method with velocity profile (4) yields an exponent $\alpha(X)$ which is too low to permit accurate approximation of the velocity profile by the correction terms used. A fixed (high) exponent might give good results, but corresponding calculations often break down for obscure reasons. On the whole, this approach was found to be troublesome and unreliable, and despite considerable effort and some successes, no standard computing technique could be developed. The power series method could be used if only overall characteristics of the boundary layer are sought. Rapid estimates will result for $N = 1$ or 2.

B. *Vortex and Axisymmetric Wake/Jet*

The power series method was first used to calculate quasicylindrical vortex flows for different swirl values [11]. The integral relations (2) for quasicylindrical vortex flow

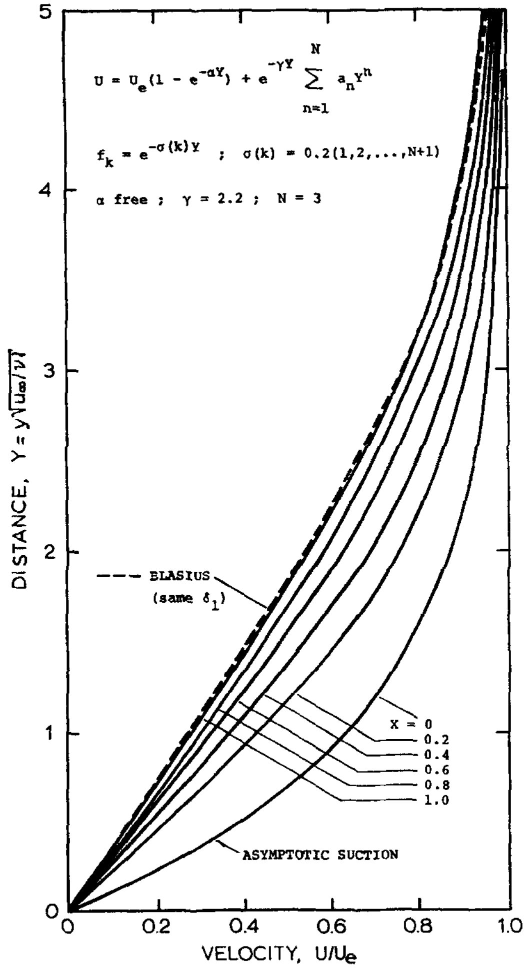

FIG. 4. Transition from the asymptotic suction to the Blasius profile by the modified power series method.

are analytically integrated in the Y direction by taking the velocity approximation,

$$U(X, Y) = U_e(X) + e^{-\alpha(X)Y} \sum_{n=0}^{N} a_n(X)\, Y^n,$$

where $a_0(X)$ is now left free to adjust itself and permit nonzero velocity on the axis; and the circulation approximation,

$$K(X, Y) = K_e(X) + e^{-\beta(X)Y} \sum_{n=0}^{N} b_n(X)\, Y^n,$$

where $b_0(X) = -K_e(X)$ to enforce solid rotation at the axis, as required in viscous vortex flow. The weighting function sets were $\{g_k(Y)\} = \{f_k(Y)\} = \{e^{-kY}\}$.

Results for a typical case and $N = 2$ are presented in Fig. 5. These results were later checked by the exponential series method for $N = 2$. Results from the two methods agree quite well, with a maximum difference of about 3 %.

The computational experience with this method was similar to that for the boundary layer flow. Although large gradients exist near the vortex breakdown point, the program often calculated through the breakdown point into the reversed flow region. More details of the application of the power series method to vortex flows are reported in [11].

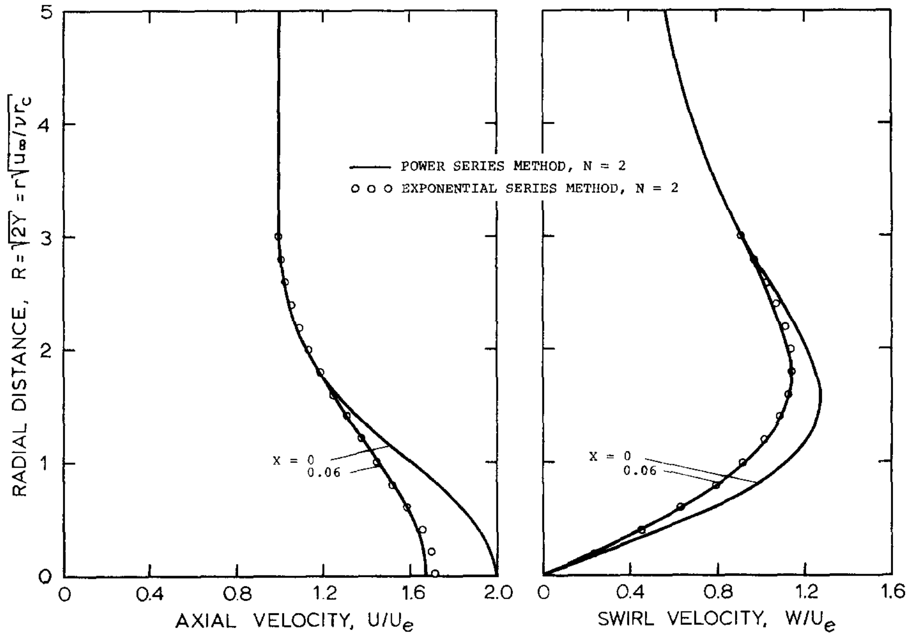

FIG. 5. Vortex velocity profil developement by the power series and exponential series methods.

V. Application of the Exponential Series Method

A. *Boundary Layer Flow*

In the exponential series method, the velocity approximation, (Eq. 5)

$$U(X, Y) = (1 - e^{-\alpha Y}) \left(U_e(X) + \sum_{n=1}^{N} a_n(X)\, e^{-n\alpha Y}\right),$$

(α fixed) and the weighting function set, $f_k(Y) = e^{-\sigma(k)Y}$, are substituted into the general integral relations for the boundary layer Eqs. (1). This leads to N first-order ordinary differential equations for the parameters $a_n(X)$. They have the form

$$\sum_{n=1}^{N} \dot{a}_n C_{n,k} = D_k , \qquad k = 1, 2,..., N. \tag{6}$$

TABLE II

Nondimensional Wall Shear, Displacement and Momentum Thicknesses, and Separation Point for Flow Over a Circular Cylinder.
Dimensional values for $U = 2\sin(x/R)$:

$$\tau_w = (\rho U_\infty^2 \sqrt{8}/\sqrt{Re_\infty})\, T_w ,$$

$$\delta_1 = (R/\sqrt{Re_\infty}\,\sqrt{2})\, \Delta_1 , \qquad \delta_2 = (R/\sqrt{Re_\infty}\,\sqrt{2})\, \Delta_2 .$$

x/R	Integral Method $N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$	Terrill
0.3	T_w = 0.36114	0.35810	0.35685	0.35669	0.35686	0.3569
	Δ_1 = 0.88898	0.70331	0.66239	0.65270	0.65444	0.6595
	Δ_2 = 0.45888	0.34252	0.30034	—	0.29210	0.2971
0.5	0.56106	0.55985	0.55750	0.55751	0.55755	0.5575
	0.91486	0.73161	0.68709	0.67132	0.67616	0.6813
	0.46920	0.35794	0.31247	—	0.30111	0.3061
0.8	0.74721	0.75965	0.75527	0.75516	0.75508	0.7550
	0.97919	0.80794	0.75485	0.72701	0.73668	0.7408
	0.49292	0.39779	0.34551	—	0.32633	0.3303
1.0	0.77440	0.80386	0.79895	0.79823	0.79802	0.7979
	1.03985	0.88995	0.82882	0.78929	0.80369	0.8057
	0.51275	0.43760	0.38093	—	0.35397	0.3560
1.2	0.71764	0.76592	0.76396	0.76147	0.76113	0.7611
	1.11501	1.00995	0.93859	0.88494	0.90516	0.9031
	0.53393	0.48983	0.43151	—	0.39498	0.3927
1.5	0.49285	0.53383	0.56107	0.55052	0.55159	0.5520
	1.25296	1.32595	1.22870	1.15803	1.18427	1.1685
	0.56299	0.59056	0.54860	—	0.49940	0.4826
1.6	0.39128	0.39024	0.45109	0.43531	0.43901	0.4396
	1.30427	1.51849	1.39362	1.32506	1.34692	1.3241
	0.57056	0.62470	0.60174	0.52212	0.55283	0.5281
1.78	0.19541		0.15232	0.11245	0.15597	0.1540
	1.40012		2.06510	2.04425	1.91493	1.8916
	0.58001		0.70531	0.67719	0.68607	0.6455
1.8	0.17362		(0.07850)		0.10914	0.1049
	1.41086		(2.44765)		2.04538	2.0326
	0.58068		(0.70442)		0.70400	0.6628
x_s/R	(1.967)	(1.675)	(1.801)	(1.805)	1.8252	1.823

The coefficients $C_{n,k}$ and R_k appear as

$$C_{n,k} = \sum_{l=0}^{N} a_l P(k, n, l)$$

and

$$D_k = \sum_{l=0}^{N} a_l Q(k, l),$$

where $a_0(X) = U_e(X)$. The coefficients P and Q are functions of n, l, the constant weighting function exponents $\sigma(k)$, and the constant exponent α. Thus they must only be determined once at the beginning of the calculation, in contrast to the power series method where the changing $\alpha(X)$ requires recomputation of similar coefficients at each step. Computation of the coefficient matrices $[C_{n,k}]$ and $[D_k]$ in the exponential series method therefore turns out to be a very efficient operation. Since this step is repeated several thousand times in a typical calculation, the advantage is obvious.

The first-order system (6) is solved for the $\dot{a}_n$ by Gaussian elimination, and the system for the $a_n(X)$ is then integrated by standard numerical methods for the solution of ordinary differential equations. With few exceptions a fourth-order Runge–Kutta technique was used. Typical boundary layer calculations take from about 30 sec (for $N = 2$) to 5 min (for $N = 5$) on a IBM 360/75 computer.

Table II presents results for the circular cylinder for $\alpha = 1$ and $N = 1$ to 5. The calculations were started at $X = 0.1$ with the stagnation point profile. The table

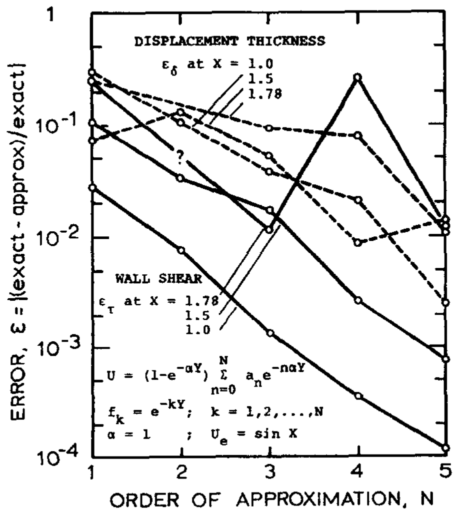

FIG. 6. Error in wall-shear and displacement thickness by the exponential series method for the boundary layer on a circular cylinder.

compares results for wall shear, displacement thickness and momentum thickness for the different N to the results computed by Terrill [12]. It is seen that these characteristics of the boundary layer are accurately computed. The apparent convergence of these quantities for increasing N is shown in Fig. 6. The question remains of whether the solution converges to the correct velocity profile, and Fig. 7 compares velocity profiles at $X = 1.0$, 1.5, and 1.78 obtained for different N to those computed by Terrill. It is interesting to note that for $N = 1$ the program integrated right through the separation point into the separated region. At higher N large gradients cause failure of the calculation very close to the separation point.

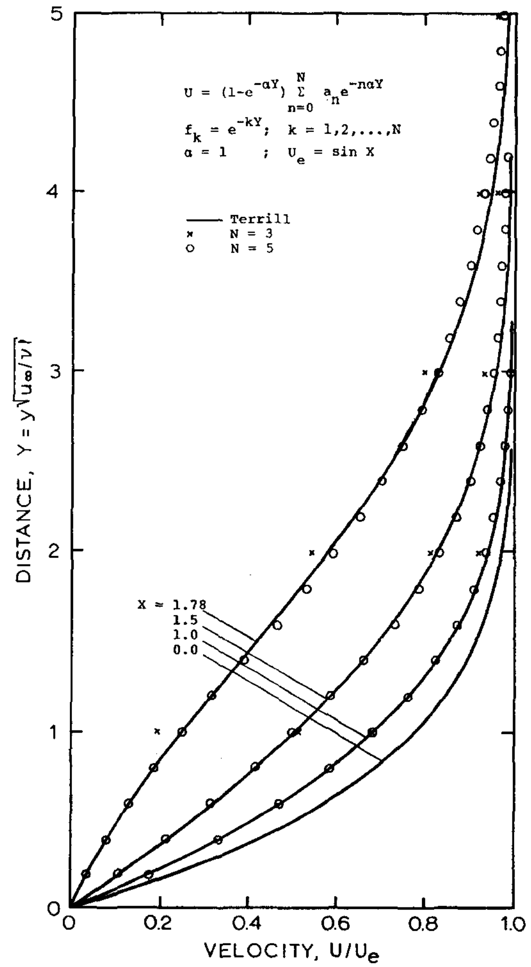

FIG. 7. Velocity profiles on the circular cylinder by the exponential series method.

The results point to convergence of the method to the exact result as N is increased, but no analytical and general proof exists presently. Another observation supporting convergence is the fact that for given N, and a given similarity profile, both the initial profile approximation (which is convergent as explained earlier) and the full integral relation routine appear to result in the same parameters (minor differences exist, but these are probably the result of the numerical integration in the initial profile routine). In contrast, completely different parameters resulted in the power series method.

Unlike the power series method, application of the exponential series method has been found to be virtually troublefree. Singular behavior does not appear, except where the boundary layer equations themselves break down, as at the flat plate leading edge or at the separation point. As usual, smaller step sizes have to be taken where gradients are large, as near the flat plate leading edge. This leads to increases in computing time there, especially for higher orders N of approximation. Discontinuous external flow parameters (discontinuous suction, discontinuous pressure gradient) cause no problems.

In the plane wake/jet calculations, one more free parameter $U_{ax}(X)$ is added by using the velocity approximation,

$$U(X, Y) = (1 - e^{-\alpha Y}) \left[U_e(X) + \sum_{n=1}^{N} a_n(X)\, e^{-n\alpha Y} \right] + e^{-\alpha Y} U_{ax}(X),$$

where U_{ax} is the velocity on the axis.

B. *Vortex and Axisymmetric Wake/Jet*

The exponential series method was applied to the integral formulation of quasi-cylindrical vortex flow Eqs. (2). These relations were integrated by applying the velocity approximation,

$$U(X, Y) = (1 - e^{-\alpha Y}) \left[U_e(X) + \sum_{n=1}^{N} a_n(X)\, e^{-n\alpha Y} \right] + U_{ax}(X)\, e^{-\alpha Y},$$

the circulation approximation,

$$K(X, Y) = (1 - e^{-\alpha Y}) \left[K_e(X) + \sum_{n=1}^{N} b_n(X)\, e^{-n\alpha Y} \right],$$

and the weighting function sets,

$$f_k(Y) = e^{-kY}, \qquad k = 1, 2, \ldots, N + 1,$$
$$g_l(Y) = e^{-lY}, \qquad l = 1, 2, \ldots, N.$$

In the N-th order approximation a system of $(2N + 1)$ first-order ordinary differential equations results for the parameters U_{ax}, a_n and b_n. The equations are integrated in the same manner as the boundary layer case. In particular, the coefficient matrices again depend mainly on numbers which are determined once at the beginning of the calculation.

Results of computations with $\alpha = 1$ and $N = 2, 3, 4$ for a vortex of medium swirl are shown in Fig. 8, which depicts its development in a zero pressure gradient.

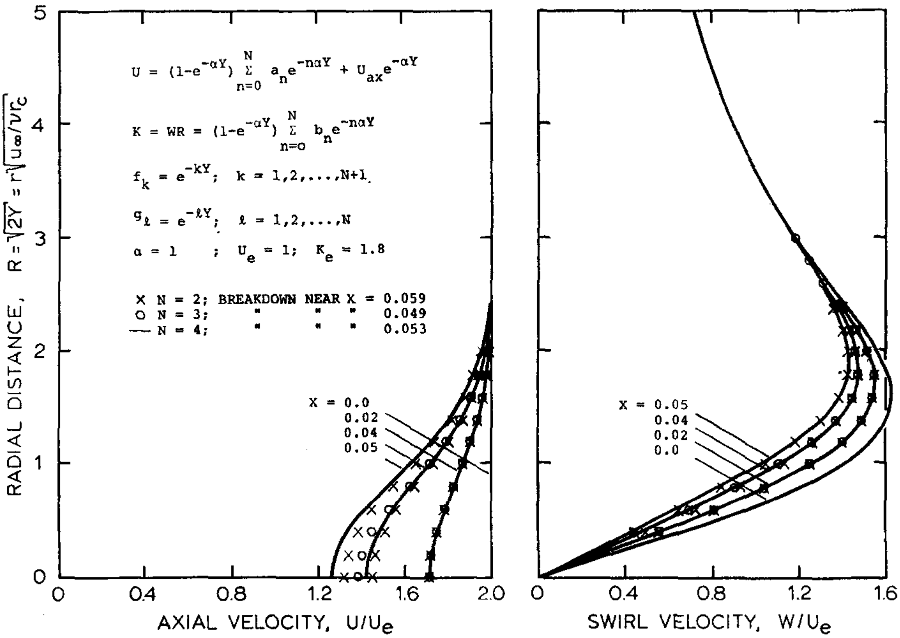

FIG. 8. Vortex velocity profile development by the exponential series method for initial uniform axial velocity.

In interpreting the results one should keep in mind the coordinate transformation introduced earlier in the vortex equations. For a typical core Reynolds number of 10^4, a nondimensional distance $\Delta x = 0.01$ corresponds to a physical distance of the order of the core thickness.

The vortex computations proceed smoothly and troublefree except in those cases where large initial swirl values, or adverse pressure gradients, or both, cause vortex breakdown. In these situations the quasicylindrical vortex equations are no longer valid, as shown in [13]. Typical computing times were of the order of 20 sec ($N = 2$) to 1 min ($N = 4$) on the IBM 360/75, i.e., shorter than for the average boundary layer program by the exponential series method on account of lesser gradients and larger possible step sizes. Exact checks of the method were not possible due to lack of published solutions. Hall's [14] calculations cannot be duplicated since he prescribes the shape of an outer stream surface, while the present method treats the free vortex. Where results can be compared, they appear to agree. A second independent check is by comparison with the results of the polynomial method. The close agreement was noted earlier. Finally, the apparent convergence of the solutions for increasing N and the success of the formulation in the boundary layer case are reassuring.

VI. Summary and Conclusions

Integral methods for the computation of laminar incompressible plane boundary layer and wake–jet flows, and for vortex and axisymmetric wake/jet flows were formulated, and typical results of computations were presented. These methods used exponentials in both the approximating and weighting functions. This choice was based on the known exponential behavior of the solutions and on the simplicity, analytical integrability, and convergence of resulting integrals.

Two different approaches were developed. One, the power series method, uses the formulation,

$$e^{-\alpha(X)Y} \sum_n a_n(X)\, Y^n,$$

in the velocity approximation. The other, exponential series method, employs the sum,

$$\sum_n a_n(X)\, e^{-\alpha n Y},$$

in the formulation. Both approximations can handle all physically possible flow profiles, including those with overshoot and flow reversal. The properties of these approximations were discussed, a method of profile approximation was described, and results were presented. Both methods can approximate typical velocity profiles (Blasius, stagnation point, and separation), and an extreme case (ramp profile) quite well with very small error even for low orders of approximation. Furthermore, the approximations appear to converge as N increases. However, on theoretical grounds, objections were raised to the power series formulation, while the exponential series approximation was shown to amount to series expansion (in a transformed finite region) of the velocity profiles in terms of shifted Legendre polynomials. Convergence for $N \to \infty$ then follows for continuous velocity profiles by Weierstrass' approximation theorem in the exponential series case.

General integral relations for plane incompressible boundary layer and wake/jet flows, and for quasicylindrical vortex and axisymmetric wake/jet flows were given. Substitution of the power series and exponential series approximations resulted in two distinct methods of computation. The power series method was found to give good overall results (wall shear, displacement and momentum thicknesses), but it converges to incorrect detailed solutions. Computations with this method were often troublesome.

The integral methods based on the exponential series approximation were found to give accurate, and apparently convergent, results. No computational problems were encountered with these methods except where the partial differential equations were not appropriate anyway. On the basis of present experience it is felt that the exponential series integral method can be used as a reliable standard method for boundary layer computation. Low orders of approximation ($N = 1$ or 2) will give good engineering estimates at the expense of very little computer time, while higher orders of approximation ($N = 3$ to 5) yield solutions approaching the accuracy of finite difference methods at a possible saving in computing time. At present the programs are not optimized with respect to computing time. The velocity approximation (5) can be rewritten in a form where parameters remain constant in similar flows. Large gradients would be avoided, and computing times reduced. The integral methods presented require no iterations, and stability problems, if any, are limited to those of the routine used for integration of the ordinary differential equations.

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